The maintenance of unidirectionally patrolled stations

Brian D. Bunday and Ruzela Tapsir

University of Bradford, Bradford, UK

This paper deals with the problem of the maintenance of N identical stations that are looked after by one operative who patrols the stations in the order $1, 2, \ldots, N, 1, 2, \ldots$, (Figure 1). The time to travel from one station to the next, $i \to i + 1$ for $i = 1, 2, \ldots, N - 1$, or $N \to 1$, is a random variable R_i that is identically and independently distributed for each i. If a station is found to be in need of repair when the operative visits it an additional time B_i is needed to carry out this work. The B_i's are independent and identically distributed random variables for each i. In addition only a certain proportion of repairs are successfully executed; σ is the probability that a repair attempt is successful. It is assumed that each station fails randomly at average rate λ in running time.

This paper shows how the important characteristics of the system can be computed from the solution of a set of equations involving σ and the Laplace transforms of R_i and B_i.

Keywords: availability, efficiency, computer performance, polling model, unidirectional polling

1. Introduction

The first attempts to model the system of N identical stations looked after by a patrolling operative date back to 1957 to the work of Runnenberg.[1] In this same year Mack, Murphy, and Webb[2] presented the problem in the realistic context of textile winding with the assumption of constant travel time between each machine (station) and constant repair time. Mack[3] extended the results to the case of variable repair time. His analysis is difficult and complex. In these papers and subsequent applications, it was assumed that the machines were identical and broke down randomly at an average rate λ in running time, so that running times for each machine have a negative exponential distribution with mean $1/\lambda$.

The attempt to model modern automatic winding machinery in the textile industry by Bunday and El-Badri[4] was the first time unsuccessful repair attempts were taken into account. The case of heterogeneous machines was discussed by Bunday and Khorram,[5] also in the context of the textile industry. Of course applications of the problem without a patrolling element had been considered much earlier (see, for example, Ashcroft[6] and Benson and Cox[7]). Again these models were developed with industrial applications very much in mind.

More recently interest in these models has been much revived because the same mathematical formulations arise in the context of computer performance evaluation. The machines correspond to computer terminals that are serviced by a central processor (the operative). This scans the terminals unidirectionally for messages (programs or information). A terminal free of messages corresponds to a running machine. A terminal that has a message in its (one) buffer corresponds to a machine that is broken down, and the time to transmit the message to the central processor corresponds to the repair time. It is assumed that messages arrive at each free terminal at random at average rate λ and that messages that arrive at a terminal

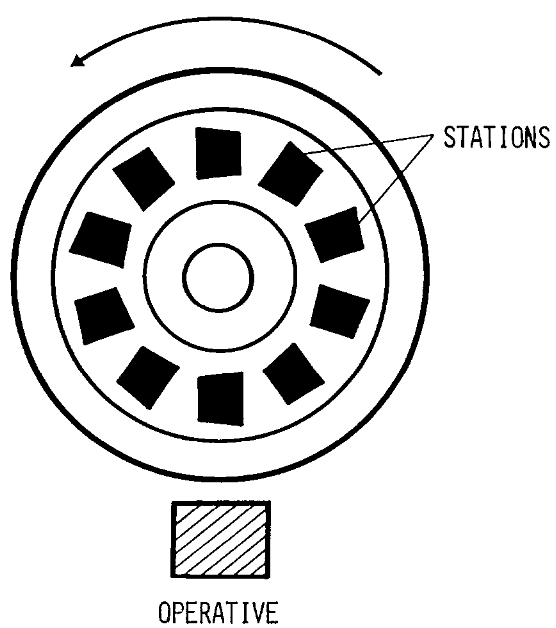

Figure 1. *N* identical stations.

Address reprint requests to Dr. Bunday at the University of Bradford, Dept. of Mathematics, Bradford, West Yorkshire BD7 1DP, UK.

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with a full buffer are lost to the system. The time for the central processor to "switch over" from one terminal to the next corresponds to the travel time or walking time of the operative between adjacent stations. When the terminals are visited in a unidirectional patrol we have what is sometimes referred to as a polling model. Takagi[8] presents an excellent review of such models and their applications and Takagi[9] also presents an extensive and up-to-date bibliography of the literature in the area.

In this context the phenomenon of unsuccessful repairs can be regarded from two viewpoints. The first is that an unsuccessful repair corresponds to an error-prone transmission system. The probability that a transmission attempt is successful is σ. In the second it is assumed that each message comprises a (variable) number of segments. At each visit to the terminal just one segment is transmitted. If the number of segments in a message has a geometric distribution with mean $1/\sigma$, the probability that the transmission of a segment corresponds to the transmission of the entire message, i.e., this segment is the final segment of the message, is σ. Either way the same mathematical model results.

2. Model assumptions and some earlier results

This model assumes that the stations are machines that break down at random, but the alternative viewpoint that they are computer terminals visited unidirectionally by a central processor, as discussed at the end of the previous section, leads to the same mathematical problem.

Thus we consider N identical stations patrolled by one operative in the order $1, 2, 3, \ldots, N, 1, 2, \ldots$, etc. Each station fails randomly at average rate λ in running time. The time for the operative to move from one station to the adjacent station is a random variable with mean r. This variable travel time includes routine inspection and maintenance that must be carried out at all stations. The time to (attempt to) repair a stopped station is a random variable with mean b. The probability that a repair attempt is successful is σ. In the case of an unsuccessful repair attempt the station concerned remains stopped at least until the next visit and repair attempt by the operative.

The problem as described above with constant travel time r between adjacent stations and constant repair time b has been considered by a number of researchers as has already been mentioned. Some of their results are given below. They are intuitively what one would expect, and their extension to the case of variable travel and repair time has been justified by Takagi.[9]

A complete patrol of the stations begins with the start of the visit to station 1 and ends with the completion of the travel time from station N to station 1. C is the random variable, which represents the duration of this patrol, and Q represents the number of stopped stations found on such a patrol. The probability that a station is found stopped on a patrol is α, and this is of course the long-term proportion of stations found stopped on a patrol. Then

$$E[C] = Nr + bE[Q] \tag{1}$$

and because

$$\alpha = \frac{E[Q]}{N} \tag{2}$$

$$E[C] = Nr + b\alpha N \tag{3}$$

A key result from the paper by Bunday and El-Badri,[4] which is *only* valid for constant travel and repair times, is that the probability of finding j machines stopped on a complete patrol is independent of which particular j machines are found stopped and only depends on the number j. They were thus able to obtain the important closed-form result

$$E[Q] = \frac{N\sum_{n=0}^{N-1}\binom{N-1}{n}\sigma^{-(n+1)}\prod_{j=0}^{n}[e^{\lambda(Nr+jb)}-1]}{1+\sum_{n=1}^{N}\binom{N}{n}\sigma^{-n}\prod_{j=0}^{n-1}[e^{\lambda(Nr+jb)}-1]}. \tag{4}$$

3. The model and its solution

In a very difficult and complex paper, Mack[3] considered the case of constant walking time and variable repair time. In the context of computer performance modelling, Takagi[8,10] considered the case of variable switchover time (travel time) and variable service time (repair time). An earlier closed-form solution to the problem of constant travel time and variable repair time by Bharucha-Reid[11] gave

$$E[Q] = \frac{N\sum_{n=0}^{N-1}\binom{N-1}{n}\prod_{j=0}^{n}\{e^{N\lambda r}[B^*(\lambda)]^{-j}-1\}}{1+\sum_{n=1}^{N}\binom{N}{n}\prod_{j=0}^{n-1}\{e^{N\lambda r}[B^*(\lambda)]^{-j}-1]\}}. \tag{5}$$

where $B^*(s)$ is the Laplace transform of the repair time probability density function. Although (5) reduces to (4) in the case $\sigma = 1$ and $B^*(s) = e^{-sb}$ (constant repair time of duration b) it is incorrect, as has been pointed out by Takagi[8] and Coffman and Gilbert.[12] The model to be described generalizes and extends the previous cases and includes the case of unsuccessful repair attempts or error-prone transmission.

We consider a system of N stations where the travel time between adjacent stations has distribution function $R(x)$ with mean r and the repair time at each station has distribution function $B(x)$ with mean b.

We let

$$p_i(v_1, v_2, \ldots, v_N) \tag{6}$$

be the probability that station v_j is in the state j at the instant when station i is visited. Here

$v_j = 1$ if station j is running (does not have a message)

$v_j = 0$ if station j is failed (has a message) $\quad$ (7)

We consider the possible events that can happen between the commencement of the visit to station i and the commencement of the visit to station $i + 1$, and because all stations are identical we consider in particular station 1 and station 2.

If station 1 is found running ($v_1 = 1$) then the interval until the operative arrives at station 2 comprises one travel time and during this period station j can fail, provided it was running initially. Of course if it was failed initially it must remain so. If station 1 is found stopped and in need of repair ($v_1 = 0$) then the duration of the interval above comprises one travel time plus one (attempted) repair time. Thus we obtain the following steady-state probability transition transform equation

$$\sum_{v_1=0}^{1} \sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} P_2(v_1, v_2, \ldots, v_N) \prod_{j=1}^{N} (z_j)^{v_j} = \sum_{v_2=0}^{1} \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(1, v_2, \ldots, v_N)$$
$$\times \int_0^\infty [(1 - e^{-\lambda x})z_1^0 + z_1 e^{-\lambda x}] \prod_{j=2}^{\infty} [(1 - e^{-\lambda x})z_j^0 + z_j e^{-\lambda x}]^{v_j} \, dR(x) + \sum_{v_2=0}^{1} \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(0, v_2, \ldots, v_N)$$
$$\times \int_0^\infty \int_0^\infty [(1 - \sigma + \sigma(1 - e^{-\lambda x}))z_1^0 + z_1 \sigma e^{-\lambda x}] \prod_{j=2}^{\infty} [(1 - e^{-\lambda(x+y)})z_j^0 + z_j e^{-\lambda(x+y)}]^{v_j} \, dR(x) \, dB(y) \tag{8}$$

where $z_1, z_2, \ldots, z_N$ are transform variables.

The joint generating function for the $P_1(v_1, v_2, \ldots, v_N)$ is defined (for $v_1 = 0, 1$) by

$$F(v_1; z_2, z_3, \ldots, z_N) = \sum_{v_2=0}^{1} \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(v_1, v_2, v_3, \ldots, v_N) \prod_{j=2}^{N} (z_j)^{v_j} \tag{9}$$

We note that because all stations are identical (i.e., we have a completely symmetrical system) that

$$P_2(v_1, v_2, v_3, \ldots, v_N) = P_1(v_2, v_3, \ldots, v_N, v_1) \tag{10}$$

for all values of $(v_1, v_2, v_3, \ldots, v_N)$.

The left-hand side of (8) can then be written as

$$\sum_{v_1=0}^{1} \sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(v_2, v_3, \ldots, v_N, v_1) \prod_{j=1}^{N} (z_j)^{v_j} = \sum_{v_3=0}^{1} \sum_{v_4=0}^{1} \cdots \sum_{v_N=0}^{1} \sum_{v_1=0}^{1} P_1(0, v_3, \ldots, v_N, v_1) z_1^{v_1} \prod_{j=3}^{N} (z_j)^{v_j}$$
$$+ \sum_{v_3=0}^{1} \sum_{v_4=0}^{1} \cdots \sum_{v_N=0}^{1} \sum_{v_1=0}^{1} P_1(1, v_3, \ldots, v_N, v_1) z_2 z_1^{v_1} \prod_{j=3}^{N} (z_j)^{v_j}$$
$$= F(0; z_3, \ldots, z_N, z_1) + z_2 F(1; z_3, \ldots, z_N, z_1)$$
$$= F(0; z_3, \ldots, z_N, z_1) + (z_2 - 1)F(1; z_3, \ldots, z_N, z_1) + F(1; z_3, \ldots, z_N, z_1). \tag{11}$$

On the right-hand side of (8), having factored out $(z_1 - 1)$ and using the joint generating function, we obtain

$$\int_0^\infty F(1; 1 - e^{-\lambda x} + z_2 e^{-\lambda x}, \ldots) \, dR(x) + (z_1 - 1) \int_0^\infty e^{-\lambda x} F(1; 1 - e^{-\lambda x} + z_2 e^{-\lambda x}, \ldots) \, dR(x)$$
$$+ \int_0^\infty \int_0^\infty F(0; 1 - e^{-\lambda(x+y)} + z_2 e^{-\lambda(x+y)}, \ldots) \, dR(x) \, dB(y)$$
$$+ \sigma(z_1 - 1) \int_0^\infty \int_0^\infty e^{-\lambda x} F(0; 1 - e^{-\lambda(x+y)} + z_2 e^{-\lambda(x+y)}, \ldots) \, dR(x) \, dB(y) \tag{12}$$

We can also express the left-hand side of (9) in the form

$$F(v_1; z_2\ z_3, \ldots, z_N) = \sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} f(v_1, v_2, \ldots, v_N) \prod_{j=2}^{N} (z_j - 1)^{v_j} \tag{13}$$

for $v_1 = 0, 1$, and this will define the functions $f(v_1, v_2, \ldots, v_N)$.

Expressed in terms of these functions the right-hand side of (11), which is of course the left-hand side of (8), becomes

$$\sum_{v_3=0}^{1} \sum_{v_4=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, v_3, \ldots, v_N, 0) \prod_{j=3}^{N} (z_j - 1)^{v_j} + \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, v_3, \ldots, v_N, 1)(z_1 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j}$$
$$+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, v_3, \ldots, v_N, 0)(z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} + \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, v_3, \ldots, v_N, 1)(z_1 - 1)(z_2 - 1)$$
$$\times \prod_{j=3}^{N} (z_j - 1)^{v_j} + \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, v_3, \ldots, v_N, 0) \prod_{j=3}^{N} (z_j - 1)^{v_j} + \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, v_3, \ldots, v_N, 1)(z_1 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \tag{14}$$

Equation (12), the right-hand side of (8), when expressed in terms of the functions $f(v_1, v_2, \ldots, v_N)$ and taking into account that

$$\int_0^\infty e^{-\lambda x \sum_{j=2}^{N} v_j}\, dR(x) = R^*\left(\lambda \sum_{j=2}^{N} v_j\right)$$

and

$$\int_0^\infty e^{-\lambda y \sum_{j=2}^{N} v_j}\, dB(y) = B^*\left(\lambda \sum_{j=2}^{N} v_j\right)$$

where $R^*(s)$ and $B^*(s)$ are the Laplace transforms of the travel time and repair time, respectively, becomes

$$\begin{aligned}
&\sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, 0, v_3, \ldots, v_N) R^*\left(\lambda \sum_{j=3}^{N} v_j\right) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, 1, v_3, \ldots, v_N) R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right](z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, 0, v_3, \ldots, v_N) R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right](z_1 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(1, 1, v_3, \ldots, v_N) R^*\left[\lambda\left(2 + \sum_{j=3}^{N} v_j\right)\right](z_1 - 1)(z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, 0, v_3, \ldots, v_N) B^*\left(\lambda \sum_{j=3}^{N} v_j\right) R^*\left(\lambda \sum_{j=3}^{N} v_j\right) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, 1, v_3, \ldots, v_N) B^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right](z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sigma \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, 0, v_3, \ldots, v_N) B^*\left[\lambda \sum_{j=3}^{N} v_j\right] R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right](z_1 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j} \\
&+ \sigma \sum_{v_3=0}^{1} \cdots \sum_{v_N=0}^{1} f(0, 1, v_3, \ldots, v_N) B^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] R^*\left[\lambda\left(2 + \sum_{j=3}^{N} v_j\right)\right](z_1 - 1)(z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j}
\end{aligned} \tag{15}$$

In (14) and (15) we equate coefficients of

$$\prod_{j=3}^{N} (z_j - 1)^{v_j},\ (z_1 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j},\ (z_2 - 1) \prod_{j=3}^{N} (z_j - 1)^{v_j}, \quad \text{and} \quad \prod_{j=1}^{N} (z_j - 1)^{v_j}$$

in turn to obtain the following equations, which are true for all $(v_3, v_4, \ldots, v_N)$.

$$\begin{aligned}
&f(0, v_3, \ldots, v_N, 0) + f(1, v_3, \ldots, v_N, 0) \\
&= f(1, 0, v_3, \ldots, v_N) R^*\left(\lambda \sum_{j=3}^{N} v_j\right) + f(0, 0, v_3, \ldots, v_N) B^*\left(\lambda \sum_{j=3}^{N} v_j\right) R^*\left(\lambda \sum_{j=3}^{N} v_j\right)
\end{aligned} \tag{16}$$

$$\begin{aligned}
&f(0, v_3, \ldots, v_N, 1) + f(1, v_3, \ldots, v_N, 1) \\
&= f(1, 0, v_3, \ldots, v_N) R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] + f(0, 0, v_3, \ldots, v_N) \sigma B^*\left(\lambda \sum_{j=3}^{N} v_j\right) R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right]
\end{aligned} \tag{17}$$

$$f(1, v_3, \ldots, v_N, 0) = f(1, 1, v_3, \ldots, v_N) R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] + f(0, 1, v_3, \ldots, v_N) B^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] R^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] \tag{18}$$

$$\begin{aligned}
f(1, v_3, \ldots, v_N, 1) &= f(1, 1, v_3, \ldots, v_N) R^*\left[\lambda\left(2 + \sum_{j=3}^{N} v_j\right)\right] \\
&+ f(0, 1, v_3, \ldots, v_N) \sigma B^*\left[\lambda\left(1 + \sum_{j=3}^{N} v_j\right)\right] R^*\left[\lambda\left(2 + \sum_{j=3}^{N} v_j\right)\right]
\end{aligned} \tag{19}$$

There are thus 2^N equations but (16) is redundant in the case $v_3 = v_4 = \cdots = v_N = 0$. We have of course to include the normalization condition

$$\sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(0, v_2, \ldots, v_N) + \sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(1, v_2, \ldots, v_N) = F(0; 1, 1, \ldots, 1) + F(1; 1, 1, \ldots) = 1 \tag{20}$$

This becomes on using (13) since all $(z_j - 1) = 0$

$$f(0, 0, \ldots, 0) + f(1, 0, 0, \ldots, 0) = 1 \tag{21}$$

The probability that station 1 is found stopped is

$$\alpha = f(0, 0, \ldots, 0) = F(0; 1, 1, \ldots, 1) = \sum_{v_2=0}^{1} \cdots \sum_{v_N=0}^{1} P_1(0, v_2, \ldots, v_N) \tag{22}$$

Thus in fact for the system (16)...(21) we only need to find $f(0, 0, \ldots, 0) = \alpha$. Of course because of the symmetry this is the probability that a station is found stopped at the moment it is visited by the operative.

With $R^*(j\lambda) = r_j$ and $B^*(j\lambda) = b_j$ we have to solve: for $N = 2$

$$\begin{pmatrix} \sigma r_1 & -1 & r_1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & r_1 b_1 & -1 & r_1 \\ 0 & \sigma r_2 b_1 & 0 & r_2 - 1 \end{pmatrix} \begin{pmatrix} f(0,0) \\ f(0,1) \\ f(1,0) \\ f(1,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for $N = 3$

$$\begin{bmatrix} \sigma r_1 & -1 & 0 & 0 & r_1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & r_1 b_1 & -1 & 0 & 0 & r_1 & -1 & 1 \\ 0 & \sigma r_2 b_1 & 0 & -1 & 0 & r_2 & 0 & -1 \\ 0 & 0 & r_1 b_1 & 0 & -1 & 0 & r_1 & 0 \\ 0 & 0 & \sigma r_2 b_1 & 0 & 0 & -1 & r_2 & 0 \\ 0 & 0 & 0 & r_2 b_2 & 0 & 0 & -1 & r_2 \\ 0 & 0 & 0 & \sigma r_3 b_2 & 0 & 0 & 0 & r_3 - 1 \end{bmatrix} \begin{bmatrix} f(0,0,0) \\ f(0,0,1) \\ f(0,1,0) \\ f(0,1,1) \\ f(1,0,0) \\ f(1,0,1) \\ f(1,1,0) \\ f(1,1,1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $N = 4$, the system to solve is

$$\begin{bmatrix}
\sigma r_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & r_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
0 & r_1 b_1 & -1 & . & . & . & . & . & . & r_1 & -1 & . & . & . & . & . \\
0 & \sigma r_2 b_1 & . & -1 & . & . & . & . & . & r_2 & . & -1 & . & . & . & . \\
0 & 0 & r_1 b_1 & . & -1 & . & . & . & . & . & r_1 & . & -1 & . & . & . \\
0 & 0 & \sigma r_2 b_1 & . & . & -1 & . & . & . & . & r_2 & . & . & -1 & . & . \\
0 & 0 & 0 & r_2 b_2 & . & . & -1 & . & . & . & . & r_2 & . & . & -1 & . \\
0 & 0 & 0 & \sigma r_3 b_2 & . & . & . & -1 & . & . & . & r_3 & . & . & . & -1 \\
0 & 0 & 0 & 0 & r_1 b_1 & . & . & . & -1 & . & . & . & r_1 & . & . & . \\
0 & 0 & 0 & 0 & \sigma r_2 b_1 & . & . & . & . & -1 & . & . & r_2 & . & . & . \\
0 & 0 & 0 & 0 & 0 & r_2 b_2 & . & . & . & . & -1 & . & . & r_2 & . & . \\
0 & 0 & 0 & 0 & 0 & \sigma r_3 b_2 & . & . & . & . & . & -1 & . & r_3 & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & r_2 b_2 & . & . & . & . & . & -1 & . & r_2 & . \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma r_3 b_2 & . & . & . & . & . & . & -1 & r_3 & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_3 b_3 & . & . & . & . & . & . & -1 & r_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma r_4 b_3 & . & . & . & . & . & . & . & r_4 - 1
\end{bmatrix}
\begin{bmatrix} f(0000) \\ f(0001) \\ f(0010) \\ f(0011) \\ f(0100) \\ f(0101) \\ f(0110) \\ f(0111) \\ f(1000) \\ f(1001) \\ f(1010) \\ f(1011) \\ f(1100) \\ f(1101) \\ f(1110) \\ f(1111) \end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

No neat closed-form solution to these equations appears to exist, and they were solved numerically. In the case of N stations there are 4^N matrix elements but only $3(2^N) - 3 + 2^{N-1}$ of these elements are nonzero. The NAG program for the solution of sparse sets of equations is thus a convenient way in which to obtain a solution. We need of course to be able to compute the Laplace transforms of the travel and repair time distributions.

4. Other useful performance measures

$$\alpha = f(0, 0, \ldots, 0) = \frac{E[Q]}{N} \tag{23}$$

can be interpreted as the probability that a station is found stopped by the operative or as the probability that a terminal has a message to be transmitted when visited by the server.

Of course each station alternates between the "running" state and the "stopped" state with mean durations $1/\lambda$ and $E[T]$, respectively. Here $E[T]$ is sometimes called the mean response time and is defined as the mean time from the failure of a station to the completion of its repair. In the context of computer performance it is the mean time from the arrival of a message at a free terminal to the completion of its service.

Another useful measure for the system is the throughput, which measures the average number of machines repaired in a unit time, or in the computer performance context as the average number of messages serviced in a unit time. This is given by

$$\gamma = \frac{\sigma E[Q]}{E[C]} = \frac{N}{E[T] + 1/\lambda} \tag{24}$$

because for each station the duration of a *cycle of operations*, that is running time plus response time, has mean $E[T] + 1/\lambda$.

Thus from (24) on using (1) and (3) we obtain

$$E[T] = \frac{Nb}{\sigma} + \frac{Nr}{\sigma\alpha} - \frac{1}{\lambda} \tag{25}$$

If $E[W]$ is the mean time that a station remains stopped waiting for attention, then because the mean time to execute a successful repair is b/σ

$$E[W] = E[T] - \frac{b}{\sigma} \tag{26}$$

In the computer situation the proportion of time that a station is "blocked," i.e., has a message and so cannot receive another (in the single buffer case), is

$$P_B = \frac{E[T]}{E[T] + 1/\lambda} \tag{27}$$

Thus from (24) we observe that

$$\gamma = \lambda N(1 - P_B) \tag{28}$$

This shows as we might expect that γ is the proportion $(1 - P_B)$ of the total message arrival rate at all terminals. The availability of each terminal or the efficiency of each station is the proportion of time it is free of messages or the proportion of time it is running and is given by

$$A = 1 - P_B = \frac{1/\lambda}{E[T] + 1/\lambda} \tag{29}$$

We observe that all of these measures can be evaluated if the key measure α is known.

5. Some numerical results

There are 2^N independent equations in the set (16)–(19) and (21). The NAG routines FO1BRF and FO4AXF were used to solve the sparse equations that result. The values of N were restricted to be no more than 10. There are of course still four parameters, σ, λ, r, and b, remaining so that the production of comprehensive tables would be very space consuming. Instead we reproduce just a few results in *Tables 1*, *2* and *3* using the notation given earlier. The computer program is capable of obtaining results for other values of the parameters and distributions as required. We verified that the results for the constant travel/constant service

Table 1. Characteristics of unidirectionally patrolled symmetric system, exponential walking (travel) time, constant service time
$Nr = 10$, $b = 1$, $\sigma = 0.7$, $\lambda = 0.2$

N	$E[Q]$	$E[C]$	$E[T]$	P_B	γ
2	1.6786	11.6786	14.8781	0.7485	0.1006
3	2.6590	12.6590	15.4035	0.7549	0.1470
4	3.6616	13.6616	16.3205	0.7655	0.1876
5	4.6768	14.6768	17.4159	0.7769	0.2231
6	5.6990	15.6990	18.6117	0.7882	0.2541
7	6.7246	16.7246	19.8709	0.7990	0.2815
8	7.7512	17.7512	21.1728	0.8090	0.3057
9	8.7776	18.7776	22.5049	0.8182	0.3272
10	9.8027	19.8027	23.8596	0.8267	0.3465

Table 2. Characteristics of unidirectionally patrolled symmetric system, exponential walking (travel) time, rectangular service time*
$Nr = 10$, $b = 1$, $\sigma = 0.9$, $\lambda = 0.2$

N	$E[Q]$	$E[C]$	$E[T]$	P_B	γ
2	1.5950	11.5950	11.1548	0.6905	0.1238
3	2.5529	12.5529	11.3903	0.6949	0.1830
4	3.5407	13.5407	11.9968	0.7058	0.2353
5	4.5476	14.5476	12.7719	0.7187	0.2813
6	5.5668	15.5668	13.6425	0.7318	0.3218
7	6.5933	16.5933	14.5742	0.7446	0.3576
8	7.6239	17.6239	15.5481	0.7567	0.3893
9	8.6562	18.6562	16.5525	0.7680	0.4176
10	9.6885	19.6885	17.5795	0.7786	0.4429

*Standard deviation is that of an exponential distribution with the same mean.

Table 3. Characteristics of unidirectionally patrolled symmetric system, exponential walking (travel) time, rectangular service time*
$Nr = 10$, $b = 1$, $\sigma = 0.7$, $\lambda = 0.2$

N	$E[Q]$	$E[C]$	$E[T]$	P_B	γ
2	1.6733	11.6733	14.9325	0.7492	0.1003
3	2.6459	12.6459	15.4832	0.7559	0.1465
4	3.6404	13.6404	16.4112	0.7665	0.1868
5	4.6484	14.6484	17.5091	0.7779	0.2221
6	5.6649	15.6649	18.7023	0.7891	0.2531
7	6.6863	16.6863	19.9559	0.7996	0.2805
8	7.7104	17.7104	21.2509	0.8095	0.3048
9	8.7354	18.7354	22.5755	0.8187	0.3264
10	9.7603	19.7603	23.9222	0.8271	0.3458

*Standard deviation is that of an exponential distribution with the same mean.

time case agreed with the previous results for this case obtained using the formula (4).

We interpreted this as justification for the more elaborate model needed to cater to variable travel and repair time, and as validating the general accuracy of our numerical procedures and calculations.

These calculations were carried out using a wide range of parameter values and different forms for the distributions of travel time and repair time. Clearly, with the space limitations imposed on a paper such as this, it is not possible to report on every individual case that was considered. The parameter values, λ, σ, b, and r, were restricted to values that were likely to arise in the practical context of machine production problems or computer performance evaluation situations. Among the distributions considered were the negative exponential, gamma, uniform, constant, as well as symmetric and skew distributions. The results reproduced in *Tables 1, 2,* and *3* and in graphical form are typical and representative of the calculations that were performed.

A comparison of the corresponding values in *Tables 2* and *3* shows the changes attributable to different values of σ. As σ increases so the throughput increases, while correspondingly the expected response time and blocking probability decreases. This is intuitively what we would expect as σ represents the probability of a successful repair or a reliable transmission. Of course in the second model, in which messages are modelled to have a variable number of segments, the mean number of segments is $1/\sigma$ and so decreases with increasing σ. Thus, again, our computed result is in line with our expectation. This result was confirmed for all the distributions of travel and repair time that we considered.

In *Tables 1* and *3* the parameter values are the same but the distributions of travel and repair time differ (in fact it is only the latter). The values of the important performance measures are very close. This again is typical of the many calculations we performed using a variety of parameter values and distributions. We were led to the conclusion that although the performance measures vary quite markedly, with respect to the means of the distributions of travel time and service time, they do not depend critically on the shape of these distributions.

The family of Erlang distributions (widely used in queueing theory) proved to be useful in this context. The kth member of the family has probability density function

$$f_k(x) = \frac{k\mu(k\mu x)^{k-1}e^{-k\mu x}}{\Gamma(k)}; \qquad x \geqslant 0$$

has mean $\frac{1}{\mu}$ and variance $\frac{1}{k\mu^2}$ with Laplace transform $F_k^*(s) = \left(1 + \frac{s}{k\mu}\right)^{-k}$. By varying k we obtain a family of distributions, all having the same mean but differing shapes.

[As $k \to \infty f_k(x) \to \delta\left(x - \frac{1}{\mu}\right)$ the density for the constant distribution with value $\frac{1}{\mu}$, when $k = 1$ we have the negative exponential distribution.]

Again the particular values shown in *Tables 1* and *3* are similar to those obtained for other distributions with the same means. The general result is well illustrated in *Figure 2,* which shows the throughput of the system for a particular set of parameters but different distributions for the walking (travel) and service times. The rectangular distributions have mean 1 and variance 1 (i.e., they have the same mean *and* variance as an exponential distribution with mean 1). As we might expect, because congestion in queueing systems arises from variation "all other things being equal," the constant walking/constant service time situation gives the highest throughput (least mean response time) but the differences are small and quite frankly are within the accuracy underlying the

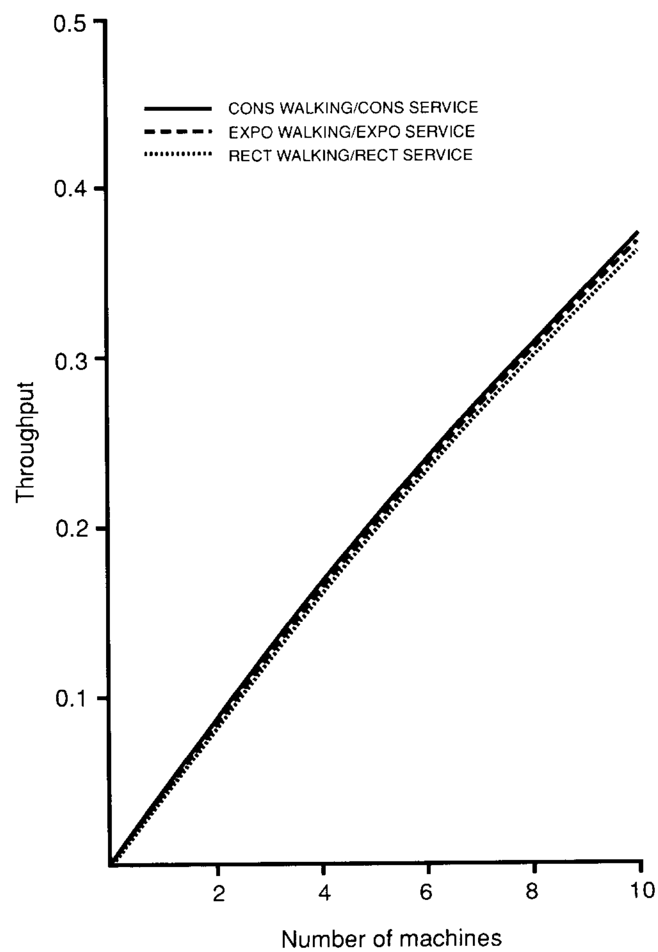

Figure 2. Comparisons of the throughput; $b = 1$, $Nr = 8$, $\sigma = 0.9$, $\lambda = 0.06$.

assumptions necessary for the whole modelling process. This was confirmed by other calculations; an upper bound to the throughput for given parameter values was provided by the constant travel/constant repair time case. Conversely, increasing the variance of either had the effect of decreasing the throughput, although, as has been stated before, the differences in any practical context were very small.

This suggests that the elaborate analysis of Section 3 is hardly necessary, for within the context of a practical application the constant travel/constant repair time solution will give an upper bound (perhaps on the optimistic side) of the true performance measures. Of course, and this is very important, for the constant travel/constant service time problem, the exact closed formula (4) holds and so makes the calculation of the characteristics for *any* value of N relatively simple.

We should perhaps be a little guarded in drawing the general conclusion above, but it did seem to be verified by the wide range of calculations that we performed; namely, the important performance characteristics do vary with the means of the distributions of travel and repair times but are very insensitive to their shape. We are fairly confident that this assertion is valid, although it may be that some researcher will rise to the challenge and present us with particular parameter values and/or distributions to contradict it. We doubt that this would constitute a reasonable and practical situation.

References

1. Runnenberg, J. T. Machines served by a patrolling operator. Mathematisch Centrum, Statistische Afdeling Report S221, Amsterdam, 1957
2. Mack, C., Murphy, T., and Webb, N. L. The efficiency of N machines unidirectionally patrolled by one operative when walking time and repair times are constants. *J. R. Statistical Soc.* 1957, **B19**, 166–172
3. Mack, C. The efficiency of N machines, unidirectionally patrolled by one operative when walking time is constant and repair times are variable. *J. R. Statistical Soc.* 1957, **B19**, 173–178
4. Bunday, B. D. and El-Badri, W. K. A model for a textile winding process. *Eur. J. Operational Res.* 1984, **15**, 55–62
5. Bunday, B. D. and Khorram, E. The efficiency of unidirectionally patrolled machines. *Appl. Math. Modelling* 1987, **11**, 380–383
6. Ashcroft, M. A. The productivity of several machines under the care of one operator. *J. R. Statistical Soc.* 1950, **B12**, 145–151
7. Benson, F. and Cox, D. R. The productivity of machines requiring attention at random intervals. *J. R. Statistical Soc.* 1951, **B13**, 65–82
8. Takagi, H. Analysis of single-buffer polling systems. IBM Tokyo Research Laboratory, Report TR87, 1988
9. Takagi, H. Queueing analysis of polling models: an update. *Stochastic Analysis of Computer and Communication Systems*, ed. H. Takagi, Elsevier Science Publishers, Amsterdam, 1990, pp. 267–318
10. Takagi, H. On the analysis of a symmetric polling system with single message buffers. *Performance Evaluation*, 1985, **5**, 149–157
11. Bharucha-Reid, A. T. Elements of the Theory of Markov Processes and Their Applications. McGraw-Hill Book Co., New York, 1960, Section 9.4D
12. Coffman, E. G. and Gilbert, E. N. A continuous polling system with constant service times. *IEEE Trans. Information Theory* 1986, **33**, 584–591